Darboux transformation for the modified Veselov-Novikov equation

2002 J. Phys. A: Math. Gen. 35 3779

(http://iopscience.iop.org/0305-4470/35/16/316)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 3779–3785 PII: S0305-4470(02)28312-2

Darboux transformation for the modified Veselov–Novikov equation

Delong Yu[1], **Q P Liu**[2] **and Shikun Wang**[3]

[1] Beijing University of Chemical Technology, Beijing 100029, People's Republic of China
[2] China University of Mining and Technology, Beijing 100083, People's Republic of China
[3] Institute of Applied Mathematics, Academy of Mathematics and System Science, Chinese Academy of Science, PO Box 2734, Beijing 100080, People's Republic of China

E-mail: yuhaijuan@sina.com.cn and wsk@amath5.amt.ac.cn

Received 20 August 2001, in final form 5 February 2002
Published 12 April 2002
Online at stacks.iop.org/JPhysA/35/3779

Abstract

A Darboux transformation is constructed for the modified Veselov–Novikov equation. By means of the Darboux transformation, two families of explicit solutions of this equation are given.

PACS numbers: 02.30.−f, 02.40.Hw

1. Introduction

Solitons and geometry are closely connected. Many soliton equations or integrable systems have their origins in classical differential geometry. The best-known example, probably the first one, is the celebrated sine–Gordon equation, which was used to describe surfaces with constant negative Gaussian curvature. Another example is the binormal flow of a curve in $\mathbb{R}^3$. It essentially appeared in the study of vortex filaments in the paper of da Rios [1]. Much later, Hasimoto [2] showed the equivalence of this system with the non-linear Schrödinger equation. For more references on the interrelation between geometry and integrable systems, we refer the reader to the recent books (see [3, 4] and the references therein).

Recently, a class of integrable deformations of surfaces immersed in $\mathbb{R}^3$ was defined by using the generalized Weierstrass representation in [5]. The main observation of that paper is that the operator from the linear problem of the generalized Weierstrass representation coincides with the operator L^{mNV} to which the modified Veselov–Novikov (mVN) hierarchy is attached. Thus, the geometrical significance of the mVN equation is established and it is important to construct the explicit solutions for this equation. It is well known that one of the most powerful techniques leading to explicit solutions for an integrable equation is Darboux transformation (DT) [8]. In this paper we construct a binary DT for the mVN equation.

This paper is organized as follows. In section 2 we will give a brief review of the mVN equation. In section 3, the derivation of the DT for the mVN equation will be given. Section 4 contains the explicit solutions of the mVN equation, which are generated by means of the DT. We give two examples: one is constructed from the simplest case: zero background $U(x, y, t) = 0$, the other from a particular solution of the MKdV equation. We present our conclusions and discussion in section 5.

2. The mVN equation and its Lax representation

The mVN equation is a natural two-dimensional generalization of MKdV equation. The MKdV equation reads as

$$U_t = \tfrac{1}{4}U_{xxx} + 6U^2U_x, \qquad (1)$$

while the mVN equation is [6]

$$U_t = (U_{zzz} + 3U_zV + \tfrac{3}{2}UV_z) + (U_{\bar{z}\bar{z}\bar{z}} + 3U_{\bar{z}}\bar{V} + \tfrac{3}{2}U\bar{V}_{\bar{z}}), \qquad V_{\bar{z}} = (U^2)_z. \qquad (2)$$

It is known that the mVN equation is represented by a Manakov triad; that is, it has the following operator formalism:

$$L_t + [L, A] - BL = 0, \qquad (3)$$

where

$$L = \begin{pmatrix} \partial & -U \\ U & \bar{\partial} \end{pmatrix},$$

$$A = \partial^3 + \bar{\partial}^3 + 3\begin{pmatrix} 0 & -U_z \\ 0 & V \end{pmatrix}\partial + 3\begin{pmatrix} \bar{V}_{\bar{z}} & 0 \\ U_{\bar{z}} & 0 \end{pmatrix}\bar{\partial} + \frac{3}{2}\begin{pmatrix} \bar{V}_{\bar{z}} & 2UV \\ -2U\bar{V} & V_z \end{pmatrix},$$

$$B = 3\begin{pmatrix} 0 & U_z \\ -U_z & 0 \end{pmatrix}\partial + 3\begin{pmatrix} 0 & U_{\bar{z}} \\ -U_{\bar{z}} & 0 \end{pmatrix}\bar{\partial} + 3\begin{pmatrix} 0 & U_{\bar{z}\bar{z}} + U(\bar{V} - V) \\ -U_{zz} - UV + U\bar{V} & 0 \end{pmatrix},$$

$$\partial = \tfrac{1}{2}(\partial_x - \mathrm{i}\partial_y), \qquad \bar{\partial} = \tfrac{1}{2}(\partial_x + \mathrm{i}\partial_y).$$

In [7], it is shown that the system (2) is related to the Veselov–Novikov equation in a similar manner to how the MKdV system is related to the KdV system.

Remarks.

(1) If the function U depends only on one space variable x, the mVN equation (2) reduces to the MKdV equation (1).
(2) The field variable U in equation (2) is assumed to be a real-valued function.

Since the mVN equation possesses the operator representation (3), it deforms the kernel of the operator L via the equation

$$\begin{aligned} L\Psi &= 0 \\ \Psi_t &= A\Psi \end{aligned} \qquad (4)$$

where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad (5)$$

is known as a wavefunction.

3. Darboux transformation of mVN equation

To construct a DT for the mVN equation, we find that it is convenient to transform the Lax pair (4) into the following form:

$$\begin{aligned}\Psi_x &= J\Psi_y + P\Psi \\ \Psi_t &= -J\Psi_{yyy} - P\Psi_{yy} + Q\Psi_y + S\Psi\end{aligned} \tag{6}$$

where

$$J = \mathrm{i}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \equiv \mathrm{i}\sigma_3, \qquad P = \begin{pmatrix}0 & 2U\\ -2U & 0\end{pmatrix} \equiv 2\mathrm{i}U\sigma_2,$$

$$Q = \begin{pmatrix}-\mathrm{i}U^2 + 3\mathrm{i}\bar{V} & \mathrm{i}U_x - 2U_y\\ \mathrm{i}U_x + 2U_y & \mathrm{i}U^2 - 3\mathrm{i}V\end{pmatrix},$$

$$S = \begin{pmatrix}(-\frac{5}{2}\mathrm{i}U_y - \frac{3}{2}U_x)U + \frac{3}{2}\bar{V}_{\bar{z}} & -2U^3 - 2U_{yy} + \frac{1}{2}U_{xx} + \frac{\mathrm{i}}{2}U_{xy} + 3U(\bar{V} + V)\\ 2U^3 + 2U_{yy} - \frac{1}{2}U_{xx} + \frac{\mathrm{i}}{2}U_{xy} - 3U(\bar{V} + V) & (\frac{5}{2}\mathrm{i}U_y - \frac{3}{2}U_x)U + \frac{3}{2}V_z\end{pmatrix},$$

and

$$\sigma_2 = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

are Pauli matrices.

We notice that the matrices J, P, Q and S have the following involution property:

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} X \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \bar{X}, \tag{7}$$

where X is one of J, P, Q, S. It is clear that if

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{8}$$

is a vector solution of (6) then

$$\Psi^* = \begin{pmatrix}-\bar{\psi}_2\\ \bar{\psi}_1\end{pmatrix} \tag{9}$$

also satisfies (6). Hence from a vector solution we obtain a matrix solution of (6):

$$\begin{pmatrix}\psi_1 & -\bar{\psi}_2\\ \psi_2 & \bar{\psi}_1\end{pmatrix}, \tag{10}$$

which we also denote by Ψ for short.

We now consider the construction of a DT for the mVN equation.

To this end, we introduce the linear system formally conjugate to (6):

$$\begin{aligned}\Phi_x &= \Phi_y J^{\mathrm{T}} + \Phi P^{\mathrm{T}} \\ \Phi_t &= -\Phi_{yyy} J^{\mathrm{T}} - \Phi_{yy} P^{\mathrm{T}} + \Phi_y Q^{\mathrm{T}} + \Phi S^{\mathrm{T}}.\end{aligned} \tag{11}$$

It is easy to see that if Ψ is a matrix solution of (6), then $\Phi = \Psi^{\mathrm{T}}$ is a matrix solution of (11).

Now with a solution Ψ of the linear system (6) and a solution Φ of the linear system (11), we introduce a 1-form

$$\begin{aligned}\omega(\Phi, \Psi) = \Phi\Psi\,\mathrm{d}y + \mathrm{i}\Phi\sigma_3\Psi\,\mathrm{d}x + \Big[&-\mathrm{i}(\Phi_{yy}\sigma_3\Psi + \Phi\sigma_3\Psi_{yy} - \Phi_y\sigma_3\Psi_y)\\ &+ \Phi_y P\Psi + \Phi P^{\mathrm{T}}\Psi_y + \Phi\begin{pmatrix}-\mathrm{i}U^2 + 3\mathrm{i}\bar{V} & \mathrm{i}U_x\\ \mathrm{i}U_x & \mathrm{i}U^2 - 3\mathrm{i}V\end{pmatrix}\Psi\Big]\mathrm{d}t,\end{aligned} \tag{12}$$

where Ψ, Φ are matrix solutions of (6), (11) respectively. It is tedious but otherwise straightforward to show that the 1-form defined above is closed, that is

Lemma 1. $d\omega(\Phi, \Psi) = 0$.

Proof. By straightforward computation. □

Thus, the following matrices:

$$\hat{\Omega}(\Phi, \Psi) = \int_{M_0}^{M} \omega = \int_{(x_0, y_0, z_0)}^{(x,y,z)} \omega, \tag{13}$$

and

$$\Omega(\Phi, \Psi) = \hat{\Omega}(\Phi, \Psi) + \begin{pmatrix} a + b\mathrm{i} & c\mathrm{i} \\ c\mathrm{i} & a - b\mathrm{i} \end{pmatrix} \tag{14}$$

are well defined, where a, b and c are real constants. In the following, we will take Ψ as a matrix solution of (6) of the form of (10).

Let Ψ_0 be any matrix solution of (6) of the form of (10), and introduce the matrices K and σ by

$$K \equiv \Psi_0 \Omega^{-1}(\Psi_0^{\mathrm{T}}, \Psi_0)\Psi_0^{\mathrm{T}} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix},$$

$$\sigma \equiv \Psi_{0y}\Psi_0^{-1} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

Lemma 2. *K and σ defined above have to be in the following forms:*

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & \bar{k}_{11} \end{pmatrix}, \qquad \sigma = \begin{pmatrix} \sigma_{11} & -\bar{\sigma}_{21} \\ \sigma_{21} & \bar{\sigma}_{11} \end{pmatrix}$$

where $k_{12} = \mathrm{i}\lambda$ and λ is a real constant.

Proof. From the involution property of J, P, Q, S and Ψ_0, we find

$$\bar{K} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{15}$$

that is, K also possesses the involution property. Meanwhile, K is symmetric:

$$K = K^{\mathrm{T}}. \tag{16}$$

So we have

$$\bar{k}_{11} = k_{22}, \qquad \bar{k}_{21} = -k_{12}, \qquad k_{12} = k_{21} = \mathrm{i}\lambda.$$

This proves the result for K. Similarly, the result for σ can be proved. □

Our DT is now conveniently formulated as

Theorem 1. *Let Ψ and Ψ_0 be the solutions of the linear system (6). Let $\Omega(\Psi_0^{\mathrm{T}}, \Psi_0)$ and $\Omega(\Psi_0^{\mathrm{T}}, \Psi)$ be given by (14) with $\Phi = \Psi_0^{\mathrm{T}}$, $\Psi = \Psi_0$ and $\Phi = \Psi_0^{\mathrm{T}}$ and $\Psi = \Psi$, respectively. Then if $\Omega^{-1}(\Psi_0^{\mathrm{T}}, \Psi_0)$ is invertible, the new matrix of wavefunctions defined by*

$$\tilde{\Psi} = \Psi - \Psi_0 \Omega^{-1}(\Psi_0^{\mathrm{T}}, \Psi_0)\Omega(\Psi_0^{\mathrm{T}}, \Psi) \tag{17}$$

satisfies (6) with the potential U, V replaced by

$$\tilde{U} = U - \lambda = U + \mathrm{i}k_{12}, \tag{18}$$

$$\tilde{V} = V + 2\mathrm{i}Uk_{12} + \bar{k}_{11}^2 - 2(\sigma_{21}k_{21} + \bar{\sigma}_{11}\bar{k}_{11}). \tag{19}$$

Proof. It is quite easy to verify that the transformed quantities do fulfil the first equation of (6). However, the verification of the second equation of (6) is too complex to do by hand. We did check the validity by means of MAPLE. □

Thus, we establish a DT for the mVN equation. It is easily seen that this DT is a binary DT.

As usual, our DT can be iterated. Let Ψ_j $(j = 1, \ldots, n)$ be n matrix solutions of the linear system (6) in the form (10). Then the new matrix wavefunction

$$\tilde{\Psi} = \Psi - \sum_{j=1}^{n} a_j \Omega(\Psi_j^{\mathrm{T}}, \Psi)$$

solves the linear system (6) with

$$\tilde{P} = P + \left[J, \sum_{j=1}^{n} \sum_{k=1}^{n} a_j \Omega(\Psi_j^{\mathrm{T}}, \Psi_k) a_k^{\mathrm{T}} \right],$$

where the a_j are given by the following linear system:

$$\sum_{j=1}^{n} a_j \Omega(\Psi_j^{\mathrm{T}}, \Psi_k) = \Psi_k, \qquad k = 1, \ldots, n.$$

4. Explicit solutions of mVN equation

In this section, we generate explicit solutions for the mVN equation. We present two examples here.

Example 1. As the first example, we generate the solutions by the above DT for the simplest case: $U(x, y, t) = 0$. Then $V(x, y, t) = 0$. The linear system (6) in this case is

$$\Psi_x = J\Psi_y, \qquad \Psi_t = -J\Psi_{yyy}. \tag{20}$$

We take

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\alpha x - \mathrm{i}\alpha y + \alpha^3 t} \\ \mathrm{e}^{\beta x + \mathrm{i}\beta y + \beta^3 t} \end{pmatrix}, \tag{21}$$

as a vector solution, where α, β are real constants. The matrix solution is found to be

$$\Psi_0(x, y, t) = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\alpha x - \mathrm{i}\alpha y + \alpha^3 t} & -\mathrm{e}^{\beta x - \mathrm{i}\beta y + \beta^3 t} \\ \mathrm{e}^{\beta x + \mathrm{i}\beta y + \beta^3 t} & \mathrm{e}^{\alpha x + \mathrm{i}\alpha y + \alpha^3 t} \end{pmatrix}; \tag{22}$$

we obtain

$$\begin{aligned} \tilde{U} = \frac{1}{\det \Omega} \Bigg[& (\mathrm{e}^{2(\alpha x + \alpha^3 t)} - \mathrm{e}^{2(\beta x + \beta^3 t)}) \left(\frac{2}{\alpha + \beta} - \frac{2}{\alpha + \beta} \cos(\alpha + \beta) y \mathrm{e}^{(\alpha+\beta)x + (\alpha^3 + \beta^3)t} + c \right) \\ & - \left(\frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\alpha} \mathrm{e}^{2(\alpha x + \alpha^3 t)} \cos 2\alpha y + \frac{1}{\beta} \mathrm{e}^{2(\beta x + \beta^3 t)} \cos 2\beta y - 2b \right) \cos(\beta - \alpha) y \\ & - \left(\frac{1}{\alpha} \mathrm{e}^{2(\alpha x + \alpha^3 t)} \sin 2\alpha y + \frac{1}{\beta} \mathrm{e}^{2(\beta x + \beta^3 t)} \sin 2\beta y + 2a \right) \sin(\beta - \alpha) y \Bigg] \end{aligned}$$

with

$$\begin{aligned} \det \Omega = \det \Omega(\Psi_0^{\mathrm{T}}, \Psi_0) = & \left[\frac{2}{\alpha + \beta} - \frac{2}{\alpha + \beta} \cos(\alpha + \beta) y \mathrm{e}^{(\alpha+\beta)x + (\alpha^3 + \beta^3)t} + c \right]^2 \\ & + \left[-\frac{1}{2\alpha} + \frac{1}{2\beta} + \frac{1}{2\alpha} \mathrm{e}^{2(\alpha x + \alpha^3 t)} \cos 2\alpha y - \frac{1}{2\beta} \mathrm{e}^{2(\beta x + \beta^3 t)} \cos 2\beta y + b \right]^2 \\ & + \left[\frac{1}{2\alpha} \mathrm{e}^{2(\alpha x + \alpha^3 t)} \sin 2\alpha y + \frac{1}{2\beta} \mathrm{e}^{2(\beta x + \beta^3 t)} \sin 2\beta y + a \right]^2 \end{aligned}$$

is a family solution of the mVN equation involving three parameters a, b, c.

Example 2. Since the MKdV equation is a dimensional reduction of the mVN equation, we may construct new solutions for the mVN equation based on the solutions of the MKdV equation. It is easy to see that

$$U(x,t) = \frac{\sin(x+\frac{t}{2})}{2\sqrt{2}(\sqrt{2}-\sin(x+\frac{t}{2}))} \tag{23}$$

is a solution of MKdV equation (1). The corresponding vector wavefunction is

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\sqrt{\sqrt{2}-\sin(x+\frac{t}{2})-\cos(x+\frac{t}{2})}\\ \sqrt{\sqrt{2}-\sin(x+\frac{t}{2})+\cos(x+\frac{t}{2})}\end{pmatrix}\frac{\mathrm{e}^{\frac{\mathrm{i}y}{2}}}{\sqrt{2}-\sin(x+\frac{t}{2})} \tag{24}$$

with $U(x,t)$ and $V = U^2$ satisfying the linear system (6). The matrix solution in the meaning of (10) is

$$\Psi_0(x,y,t) = \frac{1}{\sqrt{2}-\sin(x+\frac{t}{2})} \times \begin{pmatrix}\sqrt{\sqrt{2}-\sin(x+\frac{t}{2})-\cos(x+\frac{t}{2})}\,\mathrm{e}^{\frac{\mathrm{i}y}{2}} & -\sqrt{\sqrt{2}-\sin(x+\frac{t}{2})+\cos(x+\frac{t}{2})}\,\mathrm{e}^{-\frac{\mathrm{i}y}{2}}\\ \sqrt{\sqrt{2}-\sin(x+\frac{t}{2})+\cos(x+\frac{t}{2})}\,\mathrm{e}^{\frac{\mathrm{i}y}{2}} & \sqrt{\sqrt{2}-\sin(x+\frac{t}{2})-\cos(x+\frac{t}{2})}\,\mathrm{e}^{-\frac{\mathrm{i}y}{2}}\end{pmatrix}.$$

Now by (13) and (14), we obtain

$$\hat{\Omega}(\Psi_0^{\mathrm{T}},\Psi_0) = -2\mathrm{i}\begin{pmatrix}\frac{\mathrm{e}^{\frac{\mathrm{i}y}{2}}}{\sqrt{2}-\sin(x+\frac{t}{2})}-\frac{\sqrt{2}}{2} & -\frac{\cos(x+\frac{t}{2})}{\sqrt{2}-\sin(x+\frac{t}{2})}-\frac{\sqrt{2}}{2}\\ -\frac{\cos(x+\frac{t}{2})}{\sqrt{2}-\sin(x+\frac{t}{2})}-\frac{\sqrt{2}}{2} & -\frac{\mathrm{e}^{\frac{-\mathrm{i}y}{2}}}{\sqrt{2}-\sin(x+\frac{t}{2})}+\frac{\sqrt{2}}{2}\end{pmatrix}, \tag{25}$$

$$\Omega(\Psi_0^{\mathrm{T}},\Psi_0) = -2\mathrm{i}\begin{pmatrix}\frac{\mathrm{e}^{\frac{\mathrm{i}y}{2}}}{\sqrt{2}-\sin(x+\frac{t}{2})}+a+b\mathrm{i} & -\frac{\cos(x+\frac{t}{2})}{\sqrt{2}-\sin(x+\frac{t}{2})}+c\\ -\frac{\cos(x+\frac{t}{2})}{\sqrt{2}-\sin(x+\frac{t}{2})}+c & -\frac{\mathrm{e}^{\frac{-\mathrm{i}y}{2}}}{\sqrt{2}-\sin(x+\frac{t}{2})}-a+b\mathrm{i}\end{pmatrix}, \tag{26}$$

where a, b and c are any real constants. We use the notation

$$\begin{aligned}\Delta &\equiv \det\Omega(\Psi^{\mathrm{T}},\Psi)\\ &= 4\left[\left[\frac{\cos(x+\frac{t}{2})}{\sqrt{2}-\sin(x+\frac{t}{2})}-c\right]^2+\left[\frac{\sin y}{\sqrt{2}-\sin(x+\frac{t}{2})}+b\right]^2\right.\\ &\quad\left.+\left[\frac{\cos y}{\sqrt{2}-\sin(x+\frac{t}{2})}+a\right]^2\right],\end{aligned}$$

$$\Omega^{-1}(\Psi_0^{\mathrm{T}},\Psi_0) = \frac{2\mathrm{i}}{\Delta}\begin{pmatrix}\frac{\mathrm{e}^{\frac{-\mathrm{i}y}{2}}}{\sqrt{2}-\sin(x+\frac{t}{2})}+a-b\mathrm{i} & -\frac{\cos(x+\frac{t}{2})}{\sqrt{2}-\sin(x+\frac{t}{2})}+c\\ -\frac{\cos(x+\frac{t}{2})}{\sqrt{2}-\sin(x+\frac{t}{2})}+c & -\frac{\mathrm{e}^{\frac{\mathrm{i}y}{2}}}{\sqrt{2}-\sin(x+\frac{t}{2})}-a-b\mathrm{i}\end{pmatrix}.$$

Then we have

$$\begin{aligned}k_{12} = \frac{2\mathrm{i}}{\Delta(\sqrt{2}-\sin(x+\frac{t}{2}))^2}&\left[2\left(\sqrt{2}\sin\left(x+\frac{t}{2}\right)-1\right)\left(\frac{1}{\sqrt{2}-\sin(x+\frac{t}{2})}+a\cos y+b\sin y\right)\right.\\ &\left.-2\left(\frac{-\cos(x+\frac{t}{2})}{\sqrt{2}-\sin x}+c\right)\cos\left(x+\frac{t}{2}\right)\right],\end{aligned}$$

and

$$\tilde{U} = U(x,t)+\mathrm{i}k_{12} = \frac{\sin(x+\frac{t}{2})}{2\sqrt{2}(\sqrt{2}-\sin(x+\frac{t}{2}))}+\mathrm{i}k_{12}$$

is a double-periodic solution of the mVN equation including three constants a, b and c.

In the special case of $b = 0$, $a = 0$, we obtain a family solution of the MKdV equation (1) depending on a parameter c. If we further put $c = 0$, we have

$$\tilde{U} = -\frac{\sin(x + \frac{t}{2})}{2\sqrt{2}(\sqrt{2} + \sin(x + \frac{t}{2}))}.$$

5. Conclusions

In this paper we present a binary DT for the mVN equation. We also calculate the solutions of the mVN equation using our DT by dressing the zero-background and MKdV solutions.

Keeping in mind the geometrical background of the mVN equation, it will be interesting to construct solutions based on more sophisticated seeds and study their geometrical implications. This may be considered in a separate work.

Acknowledgments

This work was supported in part by National Natural Science Foundation of China under grant number 19971094, the Scientific Foundation of the Chinese Academy of Sciences and the Scientific Foundation of Beijing University of Chemical Technology under grant number QN0137. We should like to thank Dr Y K Lau for many useful discussions. We also thank the anonymous referees for suggestions.

References

[1] da Rios 1906 ul moto d'un liquido indefinito co un filetto vorticoso di forma qualunque *Rend. Circ. Mat. Palerno* **22** 117
[2] Hasimoto H 1972 A soliton on a vortex filament *J. Fluid Mech.* **51** 477
[3] Fordy A P and Wood J C (ed) 1994 *Harmonic Maps and Integrable Systems* (Braunschweig: Vieweg)
[4] Terng C L and Uhlenbeck K (ed) 1998 *Surveys in Differential Geometry: Integrable Systems* (Hong Kong: International)
[5] Konopelchenko B G 1996 Induced surfaces and their integrable dynamics *Stud. Appl. Math.* **96** 9
[6] Taimanov I A 1997 Modified Veselov–Novikov equation and differential geometry of surface *Trans. Am. Math. Soc.* **197** 133
[7] Bognadov L V 1987 Veselov–Novikov equation as a natural two-dimensional generalization of the Korteweg–de Vries equation *Theor. Math. Phys.* **70** 219
[8] Matveev V B and Salle M A 1991 *Darboux Transformation and Solitons* (Berlin: Springer)